ON BASIC CONCEPTS IN SOIL DYNAMICS

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In a brief communication [1], the author presented a mathematical model which was intended for describing the motion of soil-type media. In this paper a detailed evaluation of this model is given, including the related thermodynamic problems.

Soil is a disperse medium – a mixture of mineral particles, water and air. The mineral particles form a porous skeleton (usually cemented together), the pores of which are filled with water and air. If the soil is subjected to relatively small loads, the rigidity of the skeleton is sufficient to resist those loads, the medium will behave as an elastic one and its deformations and motions can be described by a model of a linear-elastic Hookean body. With increasing load, the skeleton will gradually break up, the mineral particles will become more compact (there will be a decrease in porosity) and an increasing fraction of the load will be borne by the water and the air; further increase in load will lead to fracture not only of the weaker cementing bonds but also of the basic mineral particles of the skeleton. Under these conditions the Hookean model will become inapplicable and will have to be replaced by a new one. In constructing such a model it is necessary to see that it reduces at small loads to a Hookean model, while at high loads the properties of the medium associated with the above-mentioned processes should be taken fully into consideration. In the first instance it is necessary to consider the obvious fact that although, on increasing the stresses which compress a small element of the soil, the density of an element may increase appreciably due to recompacting of the particles and their fracture, the density will decrease only insignificantly when the load is removed, in view of the irreversibility of the processes of recompacting and fracture. Therefore, the process of loading and unloading of an element should be described by different relations.

Furthermore, in the fractured soil skeleton the bonds between individual particles are basically reduced to their mutual contacts and the force interactions are represented by mutual compression and friction at the points of contact. Therefore, the magnitude of the final mutual displacement of the particles cannot influence the resulting stresses (in contrast, for instance, to materials of the rubber type in which the final deformations determine the stresses), i.e. the components of the stress tensor should not depend on the final shear deformations. The stresses should be related to the deformations of the instantaneous state, i.e. to the tensor of the rates of deformation. In addition, this relation should conserve the important property of dry friction, by means of which the force interaction is realized between the mineral particles in the case of moderate humidity. This property consists essentially in the uniformity of the law of dry friction as a function of time. This, for instance, is not the property of the relation between stresses and deformation rates in the model of a viscous Newtonian liquid. The relations pertaining to the theory of flow and plasticity of metals do have such a property.

Finally, in the formation of the components of the stress tensor there will be not only dry friction forces between the mineral particles in contact but also the forces of elasticity which occur inside the particles themselves. Therefore, generally speaking, the elastic components have to be taken into consideration in the relation between the components of the stress tensors and the strain rates.

Such considerations have been made in the Prandtl-Reuss plasticity theory [2,3], the viewpoint of which is utilized for describing the plastic shear deformation.

It can be assumed that in an isotropic medium the character of the volume deformation is determined by the mean stress (or by the hydrostatic pressure as it is referred to in the theory of solid bodies; thereby, the magnitude of the volume deformation is determined by the change in density of the medium. Therefore, as a first hypothesis, we apply the assumption of the existence of correspondence between the mean stress (pressure) $p = -1/3(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$ and the density ρ which is, however, different for cases in which an irreversible change in volume occurs with increasing p, i.e. there is a plastic volume deformation, and in which the change in volume is reversible, i.e. the volume deformation proceeds elastically. This is the most important difference between the present model of the medium and the models of various plasticity theories in which the volume deformation is always assumed as being elastic (reversible) if it is not assumed to be completely absent [2-4]. This is due to the fact that dense non-porous materials (primarily metals), for which the plasticity theories have been evolved, do not show any appreciable non-reversible volume deformations, while for soils the ability to undergo such deformations is very characteristic, due to their disperse structure. Furthermore, they are easily susceptible to irreversible shear deformations, and in this respect they do not differ in principle from metals.

1. Basic hypothesis and the complete system of mechanical equations. It is assumed that if an element of the medium is subjected to irreversible changes in volume, the following equation will be valid for interrelating the pressure with density:

$$p = f_1(\rho) \tag{1.1}$$

Irreversible changes in density (volume) will occur only on loading; it will therefore be assumed that Equation (1.1) is valid only if the necessary condition $dp/dt > 0$ is fulfilled. If, after a certain stage of increase in p, for which an irreversible volume-change took place, the pressure in the particle begins to drop, and after some such drop the pressure may increase without reaching the initial value which was reached during the irreversible deformation, so that the change in volume proceeds reversibly, the relation between the pressure and the density for this process is determined by another equation

$$p = f_2(\rho, p_*) \tag{1.2}$$

In this relation the maximum pressure p_* to which the particular particle was exposed during the previous irreversible change in volume will be contained as a parameter. By means of Equation (1.1) the density ρ_*, which corresponds to p_*, can be introduced:

$$p_* = f_1(\rho_*) \tag{1.3}$$

and instead of p_* the parameter ρ_* can be introduced in Equation (1.2). For the given particle, the parameter p_* as well as ρ_* can only increase, and this will take place only in the case of irreversible volume deformation; in the case of elastic changes in volume, the particles p_* and ρ_* will not change. Therefore, p_* (or ρ_*) can be considered as a parameter characterizing the residual (irreversible) volume deformation.

The experiments [5] and simple intuitive considerations indicate that qualitatively the functions f_1 and f_2 are of the form as shown in Fig. 1.

Figure 2 shows the same graph in the plane p, $V = 1/\rho$. In accordance with what was said in the introduction, the function (1.1) should have a initial elastic section in the range of small p-values. In the case of unloading from any point p_*, ρ_* a reversible change in volume takes place on the curve (1.3), and the point which depicts this process is displaced

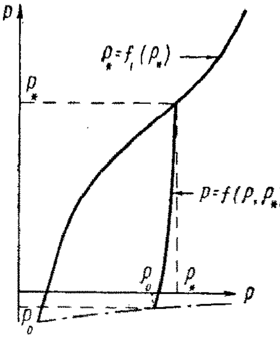

Fig. 1.

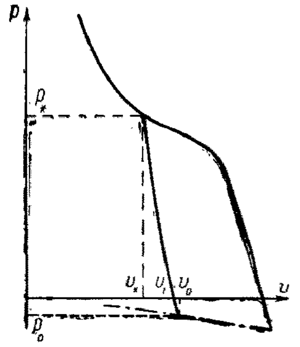

Fig. 2.

along the curve (1.2). This point can drop only to the position p_0, ρ_0, which corresponds to such a state of the element when it can no longer withstand the tensile stresses from all sides and loosens. The aggregate of such states in Figs. 1 and 2 is shown by the dashed lines, the equation of which can be written as

$$p_0 = \varphi(\rho_0) \tag{1.4}$$

This relation establishes the dependence of the loosening pressure on the density at which loosening takes place. Both these quantities are obviously determined by the degree of irreversible volume deformation, i.e. by the parameter p_* (or ρ_*). If the medium has no bonds, then $p_0 \equiv 0$. Since in a medium which in the initial natural state is bonded, the skeleton disintegrates gradually during the process of irreversible deformation and the bond decreases, with increasing p_*, i.e. ρ_0, the absolute value p_0 should decrease. From a certain value ρ_0 (i.e. p_* onwards) it can revert to zero and remain zero for all larger values of ρ_0 (p_*). This will correspond to a total loss of the bond in the case of considerable irreversible volume deformation.

In the case of very high pressures, the porosity of the medium may be completely eliminated, and the material will compress without any appreciable irreversible volume deformation in the same way as metals and non-porous rocks do. Therefore, it is necessary to take into consideration that p_* and ρ_* are bounded from above by certain limiting values $p_{*\infty}$, $\rho_{*\infty}$; once these limits are reached, Equations (1.1) and (2.2) coincide, and the entire volume deformation will become a reversible quantity. Thus, the multitude of states on the plane $p\rho$ (or p, V), which are connected by reversible changes, is depicted by a region which is bounded from above and from the left by the curve (1.3), from below by the curve (1.4) and from the right by the curve (1.2) for the value of the parameter $p_* = p_{*\infty}$. In the case that p_* (or ρ_*) are constant, the reversible changes of state in this range occur along lines of a single parameter family, which is determined by relation (1.2), the parameter on the lines of the family is p_* (or ρ_*). Outside this region reversible changes of state may occur in the case of $p > p_{*\infty}$ along the extension of the line $p_* = p_{*\infty}$, which coincides with a line (1.3) for the case that $p > p_{*\infty}$.

To simplify dealing with the fairly complicated properties of the volume deformation of the medium described above, which is particularly important for the numerical solution of problems, we will give a well-defined analytical representation of these properties. The relations which follow accomplish this aim.

In the first instance, instead of (1.1), (1.2) and (1.3), we write

$$p = f(\rho, \rho_*) e(\rho - \rho_0) e(\rho_* - \rho), \qquad p_* = f(\rho_*, \rho_*) \equiv f_1(\rho_*)$$
$$p_0 = f(\rho_0, \rho_*) \tag{1.5}$$

Furthermore, from Equations (1.4) and (1.5) we obtain

$$f(\rho_0, \rho_*) = \varphi(\rho_0) \quad \text{or} \quad \rho_0 = \Psi(\rho_*) < \rho_* \tag{1.6}$$

Finally, the condition that an irreversible volume deformation takes place only with increasing ρ_* is expressed by

$$\frac{d\rho_*}{dt} = \frac{d\rho}{dt} e(\rho - \rho_*)\, e\left(\frac{d\rho}{dt}\right) \tag{1.7}$$

In this case, the operator d/dt denotes the total time derivative. The function $e(u)$ in Equations (1.5) and (1.7) is a unit function:

$$e(u) = \begin{cases} 1 & (u \geqslant 0) \\ 0 & (u < 0) \end{cases} \tag{1.8}$$

It is obvious that, owing to relations (1.5) to (1.8), only reversible volume deformation (ρ_* = const) will occur in the case of $\rho_0 < \rho < \rho_*$, and only in the case of $\rho = \rho_*$ and $d\rho/dt > 0$ will a part of the volume deformation proceed irreversibly ($d\rho_* > 0$).

It is pointed out that while the changes in ρ_* can, generally speaking, be considerable, the changes in density in the case of purely reversible volume deformation can be small, since these are determined by the elastic deformation of the mineral particles and the water, which is insignificant. Therefore, it is valid to assume that

$$\rho_* - \rho_0 \ll \rho_* \tag{1.9}$$

This condition is essential in the sequel.

Passing to the analytical description of the shear deformation in the proposed model, it may be mentioned that we will formulate relations which in principle are apparently the most simple ones, but still incorporate all the qualitative features of shear deformations in the soil described at the beginning of the paper.

In spite of this they are mathematically very complicated. Particularly, following the basic ideas of the theory of Prandtl-Reuss, we shall assume that under conditions when the shear deformation cannot proceed purely elastically, a part of the infinitesimally small shear deformation of the instantaneous state of the element will become plastic (irreversible) and proportional to the deviator of the stress tensor. This occurs under the condition that this deviator differs from zero in a definite sense (which causes the irreversible shear deformation), whereby, generally speaking, the magnitude of this deviation, at which plastic deformation starts to occur, will depend on pressure.

The latter condition is analytically expressed in the form of a certain relation (plasticity condition), which is taken in the form of the dependence of the second invariant of this deviator on p

$$J_2 \equiv \tfrac{1}{2} S_{ij} S_{ij} = F(p) \qquad (S_{ij} = \sigma_{ij} + p\delta_{ij}) \tag{1.10}$$

F is a non-decreasing function of its argument.

The form of the plasticity condition (1.10) in the proposed model also differs from that pertaining to the model of the theory of plasticity of metals in which J_2 is assumed either constant during plastic deformation (ideal plasticity) or as depending on the characteristics of plastic deformation (hardening). Equation (1.10) is a condition of the type pertaining to ideal plasticity, in which the plasticity limit depends on the first invariant of the stress tensor, i.e. the pressure p. This is the Mises-Schleicher-type condition [3,4,6].

In the theory of the limiting equilibrium of loose granular media and soils [7,8], in which, generally, only the two-dimensional problem is considered, the condition of limiting equilibrium, i.e. the condition of plasticity, is taken as the Coulomb friction law or, more generally, the failure condition of Mohr which interrelates the normal and the shear stresses at the slip planes. These conditions, which are convenient in the case of the plane problem of limiting equilibrium, are unsuitable for the general three-dimensional case in view of the complexity of the analytical representation. Therefore, the analytically simpler relation (1.10) is proposed, which is of the same mechanical nature as the

Coulomb-Mohr condition and can be considered as being an approximation of this. In this case we follow von Mises, who in the theory of plasticity of metals substituted for Tresca's flow condition the simpler (Mises) condition, which in many cases is in better agreement with experimental data than the Tresca condition [2,3].

The correct description of the elastic part of an infinitesimally small deformation of the instantaneous state in the case that considerable deformations are possible encounters great difficulties; these are due to the fact that, on the one hand, the plastic as well as elastic components of the infinitesimally small deformation of the instantaneous state are expressed by the tensor of the deformation rates and, on the other hand, the elastic law links the stress tensor with the tensor of the elastic deformations themselves and not their rates. Therefore, to write unified relations which link the stresses with the strain rates, it is necessary to write the elasticity law in the differential form. This applies to the case of developed flow when the total deformations and displacements are not small and the state, relative to which the elastic-deformation components are considered, changes appreciably and continuously, representing a non-trivial problem.

In particular, there is difficulty in determining the rates of change of the stresses which should be used in the elastic law written in differential form. Recently, W. Prager drew attention to these problems; in a paper presented at the All-Union Congress on Mechanics he subjected to a comparative analysis various published definitions of the rates of change with time of the stress tensor and pointed out certain advantages of the definition according to Jaumann [9]. Work by Sedov [10], carried out in connection with the above paper, deals with the problem of differentiating tensors with respect to time in a general treatment of the theory of finite deformations. In the light of these concepts, the elastic deformations in the flow relations were taken into account incorrectly in our earlier work [1], and this inaccuracy is eliminated in this paper.

For describing the elastic shear deformation, Jaumann's definition of the time derivative of the stress deviator tensor [9] will be used. This leads to the following relations between the components of the stress deviator tensor and the strain deviator tensor:

$$G\left(e_{ij} - \frac{1}{3} e_{kk}\delta_{ij}\right) = \frac{\widetilde{d}S_{ij}}{dt} + \lambda S_{ij} \qquad \left(e_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (1.11)$$

where G is the shear modulus and the derivatives $\widetilde{d}S_{ij}/dt$ are determined according to Jaumann by means of the formula

$$\frac{\widetilde{d}S_{ij}}{dt} = \frac{dS_{ij}}{dt} - S_{ik}\Omega_{jk} - S_{jk}\Omega_{ik}, \qquad 2\Omega_{ij} = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \qquad (1.12)$$

The quantity λ should be positive when plastic shear deformation occurs and it should be identically zero in the case of elastic shear. Thus, relations (1.11) for $\lambda \equiv 0$ should be considered as the definition of the elastic law of the given model. It can easily be shown that in the case of small displacements and deformations this definition reduces to the ordinary Hooke's law.

As usual, the factor λ can be eliminated by means of the plasticity condition (1.10). For this purpose we multiply (1.11) by S_{ij} and take the sum

$$2GW \equiv GS_{ij}\left(e_{ij} - \frac{1}{3} e_{kk}\delta_{ij}\right) = GS_{ij}e_{ij} = S_{ij}\frac{\tilde{d}S_{ij}}{dt} + \lambda S_{ij}S_{ij} \qquad (1.13)$$

It can be shown that

$$S_{ij}\frac{\tilde{d}S_{ij}}{dt} = S_{ij}\frac{dS_{ij}}{dt} = \frac{1}{2}\frac{d\,(S_{ij}S_{ij})}{dt} = \frac{dJ_2}{dt} \qquad (1.14)$$

Therefore, (1.13) and (1.14), taking into consideration (1.10), yield for λ

$$\lambda = \frac{2GW - F'(p)\,dp\,/\,dt}{2F(p)} \qquad (1.15)$$

Equation (1.15) will be valid only for the case that $J_2 = F(p)$ and provided that $\lambda > 0$, i.e. for $2GW - F'(p)dp/dt > 0$. If these conditions are not fulfilled, then $\lambda \equiv 0$.

In the same way as above, all these properties of the parameter λ can and should be expressed by means of the single analytical expression

$$\lambda = \frac{2GW - F'(p)\,dp\,/\,dt}{2F(p)}\, e\,[J_2 - F(p)]\, e\left[2GW - F'(p)\frac{dp}{dt}\right] \qquad (1.16)$$

where $e(u)$ is the unit function (1.8). Thus, relations (1.11), (1.12) and (1.16) give a complete description of the shear deformation.

These relations, together with (1.5) to (1.8), the equations of motion

$$\rho\frac{dv_i}{dt} = \rho F_i^e - \frac{\partial p}{\partial x_i} + \frac{\partial S_{ij}}{\partial x_j} \qquad (i = 1, 2, 3) \qquad (1.17)$$

and the continuity equation

$$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0 \qquad (1.18)$$

form a complete closed system of the mechanical equations of the proposed model.

In analogy with the model of the viscous incompressible liquid, the models of the theory of plasticity and some others, the closed system of equations has been obtained without using the law of energy conservation. The problem of how the energy equations are written in such cases and, generally, what thermodynamic relations correspond to models of such a type is considered in another paper [11].

2. Thermodynamics of the medium and the energy equations. The equation of the heat inflow can be written in the general case as follows [12]:

$$\frac{\delta Q}{\delta t} = \rho \frac{d\varepsilon}{dt} - \frac{1}{2}\sigma_{ij}e_{ij} \tag{2.1}$$

where the quantity on the left-hand side of the equation represents the external heat inflow to the unit of volume per unit of time and ϵ is the internal energy per unit of mass of the medium. By using relations (1.10), (1.11), (1.13), (1.14) and (1.16), Equation (2.1) can be written as

$$\rho \frac{d\varepsilon}{dt} = \frac{\delta Q}{\delta t} - \frac{1}{2} p e_{kk} + \frac{1}{2G}\frac{dJ_2}{dt} +$$
$$+ \left[W - \frac{1}{2G}F'(p)\frac{dp}{dt}\right] e\,[J_2 - F(p)]\, e\left[W - \frac{1}{2G}F'(p)\frac{dp}{dt}\right] \tag{2.2}$$

Dividing Equation (2.2) by ρ and using relations (1.11) and (1.18), we obtain

$$\frac{d\varepsilon}{dt} = \frac{1}{\rho}\frac{\delta Q}{\delta t} - p\frac{d(1/\rho)}{dt} + \frac{1}{2G\rho}\frac{dJ_2}{dt} +$$
$$+ \frac{1}{\rho}\left[W - \frac{1}{2G}F'(p)\frac{dp}{dt}\right] e\,[J_2 - F(p)]\, e\left[W - \frac{1}{2G}F'(p)\frac{dp}{dt}\right] \tag{2.3}$$

The quantity V_e will now be introduced on the basis of the formula

$$V_e = V_1 - V, \qquad V_1 = V_1(V_*) \qquad (V = 1/\rho,\ V_* = 1/\rho_*) \tag{2.4}$$

where V_1 is the specific volume at the point of intersection of the curve of the elastic volume change with the axis of specific volumes (Fig. 2). Since, as a result of Equation (1.7), $\rho \leqslant \rho_*$, i.e. $V \geqslant V_*$ and, furthermore, $\rho \geqslant \rho_0$, i.e. $V \leqslant V_0$, the following inequalities will be valid for V_e:

$$V_1 - V_0 \leqslant V_e \leqslant V_1 - V_* \tag{2.5}$$

Furthermore, from Equations (2.4) and (1.7) it follows that

$$\frac{dV_e}{dt} = \frac{dV}{dt}\left[-1 + \frac{dV_1}{dV_*}\, e\,(V_* - V)\, e\left(-\frac{dV}{dt}\right)\right] \tag{2.6}$$

and, consequently, in the case of purely reversible volume changes, $dV_e/dt = - dV/dt$. Finally, due to Equation (1.9)

$$V_e \ll V_*, \qquad V_1 \approx V_* \tag{2.7}$$

By using the latter, it is possible to substitute V_* for V in Equation (2.3) for all cases when $1/\rho = V$, if quantities of the order of $(V_1 - V_*)/V_*$, which are small compared to unity, are neglected. As a result, this equation can be written as

$$\frac{d\varepsilon}{dt} = V_*^s \frac{\delta Q}{\delta t} - p\frac{dV}{dt} + \frac{V_*}{2G}\frac{dJ_2}{dt} +$$
$$+ V_*\left[W - \frac{1}{2G}F'(p)\frac{dp}{dt}\right] e\,[J_2 - F(p)]\, e\left[W - \frac{1}{2G}F'(p)\frac{dp}{dt}\right] \tag{2.8}$$

We will now determine the system of the thermodynamic parameters of state and the form of the thermodynamic functions of state. Let us consider at first for this purpose the reversible process. In this case $dV_*/dt = 0$, the last term on the right-hand side of (2.8) is also zero and, in accordance with the second law of thermodynamics, the external heat inflow per unit of mass, $V_*\delta Q/\delta t$, can be expressed by $T\,dS/dt$, where S is entropy per unit of mass and T the absolute temperature of the element. Thus, the equation of heat inflow (2.8) for the reversible process is transformed into the following thermodynamic identity:

$$d\varepsilon = T\,dS + p\,dV_e + \frac{V_*}{2G}dJ_2 \tag{2.9}$$

This relation indicates that the values T, V_e, J_2 have to be taken as state parameters for the reversible process. Furthermore, the thermodynamic functions may finally depend on a simple parameter (but not a parameter of state!) i.e. on the quantity V_* [11]. The conditions of integrability of dS lead to the necessity of fulfilling the following relations:

$$-\left(\frac{\partial\varepsilon}{\partial J_2}\right)_{T,\,V_e} + \frac{V_*}{2G} = T\left[\frac{\partial}{\partial T}\left(\frac{V_*}{2G}\right)\right]_{V_e,\,J_2}$$
$$-\left(\frac{\partial\varepsilon}{\partial V_e}\right)_{T,\,J_2} + p = T\left(\frac{\partial p}{\partial T}\right)_{V_e,\,J_2}, \qquad \left(\frac{\partial p}{\partial J_2}\right)_{T,\,V_e} = \left[\frac{\partial}{\partial V_e}\left(\frac{V_*}{2G}\right)\right]_{T,\,J_2} \tag{2.10}$$

The most general model for the considered parameters of state is obtained if the pressure p, shear modulus G and the specific heat $C_{V_e,\,J_2} = (\partial\epsilon/\partial T)_{V_e,\,J_2}$, with V_e and J_2 constant, are given in the form of functions of the arguments T, V_e, J_2, which satisfy only conditions (2.10) but are otherwise arbitrary. Particularly, using the well-known dependence

$$p = p(T,\ V_e,\ J_2;\ V_*) \tag{2.11}$$

we obtain, by taking into consideration (2.10)

$$\frac{V_*}{2G} = \int_0^{V_e} \left(\frac{\partial p}{\partial J_2}\right)_{T,\ V_e} dV_e + \frac{V_*}{2G_0(T,\ J_2;\ V_*)} \tag{2.12}$$

$$C_{V_e,\ J_2} = -T\left[\int_0^{V_e}\left(\frac{\partial^2 p}{\partial T^2}\right)_{V_e,\ J_2} dV_e + \int_0^{J_2}\left[\frac{\partial^2}{\partial T^2}\left(\frac{V_*}{2G}\right)\right]_{V_e,\ J_2} dJ_2\right] + C^{\circ}_{V_e,\ J_2}(T;V_*) \tag{2.13}$$

where G_0 and $C^{\circ}_{V_e,\ J_2}$ are arbitrary functions of their arguments and may depend on the parameter V_*.

From these relations, the internal energy and the entropy can be determined by integration:

$$\varepsilon = \int_0^{V_e}\left[p - T\left(\frac{\partial p}{\partial T}\right)_{V_e,\ J_2}\right] dV_e + \tag{2.14}$$

$$+\int_0^{J_2}\left\{\frac{V_*}{2G} - T\left[\frac{\partial}{\partial T}\left(\frac{V_*}{2G}\right)\right]_{V_e,\ J_2}\right\} dJ_2 + \int_0^{T} C_{V_e,\ J_2}\, dT + \varepsilon_0(V_*)$$

$$S = \int_{\varepsilon_0}^{\varepsilon}\frac{d\varepsilon}{T} - \frac{1}{T}\int_0^{V_e} p\,dV_e - \frac{1}{T}\int_0^{J_2}\frac{V_*}{2G}\,dJ_2 + S_0(V_*) \tag{2.15}$$

Here ϵ_0 and S_0 are, generally speaking, arbitrary functions of the parameter V_*, which is not a thermodynamic parameter of state since it does not change in the case of reversible processes. Determination of the form of these functions is associated with the study of the mechanism of irreversible micro-processes in the medium [11]. For computing the first integral in (2.15), which for brevity is written in symbolic notation, it is necessary to apply relation (2.10). From the known formulas, having available (2.14) and (2.15), expressions can be obtained for the remaining thermodynamic functions.

The expressions obtained for the equations of state (2.11) and (2.12), the specific heat (2.13), the internal energy and the entropy (2.14) and (2.15), depend on V_* as a parameter, i.e. on the degree of plastic volume deformation. In the course of plastic volume deformation, the paramter V_* will decrease and the process will be thermodynamically irreversible. The irreversibility will also take place in the case of shear deformation if $\lambda > 0$, and also as a result of heat conduction in the medium. In order to obtain the thermodynamic relations necessary for describing these processes, we will proceed in the same way as in the examples considered in earlier work [11]. The basic thermodynamic assumptions consist in the fact that all the thermodynamic relations written above remain valid also for irreversible processes. However, while in writing the thermodynamic equations (2.9) and (2.10) the parameter V_* is assumed constant, in substituting Equations (2.11) to (2.15) into the equations describing the processes in presence of irreversibility, it is necessary to take into consideration the changes (the decrease) in the parameter V_*, which is determined by the differential equation (1.7).

The correspondence of this assumption with reality can in the final analysis be determined only by experiment.

A simple particular case described by the above thermodynamic relations, corresponding to the mechanical model evolved in the first section of the paper, is obtained by assuming, in addition to the hypotheses which follow from (1.5) on the independence of the pressure p on T and J_2, the independence of the shear modulus G on T and J_2. In this case it follows from (2.12) that G does not depend on V_e either, and it follows from (2.13) that the specific heat depends only on T and V_*; consequently, all the specific heats will mutually coincide and the coefficient of thermal expansion will equal zero [11]. Formulas (2.14) and (2.15) will change to

$$\varepsilon = \int_0^{V_e} p(V_e; V_*)\,dV_e + \frac{V_*}{2G(V_*)} J_2 + \int_0^T C(T; V_*)\,dT + \varepsilon_0(V_*) \qquad (2.16)$$

$$S = \int_0^T \frac{C(T; V_*)}{T}\,dT + S_0(V_*) \qquad (2.17)$$

Thus, the full model proved to be a model with separable energy, whereby the entropy depends only on the temperature and V_*, which is in complete analogy to the case of an ordinary plastic medium [11].

By substituting the equation for ϵ (2.16) into Equation (2.8) and assuming for the external heat flow the law

$$\frac{\delta Q}{\delta t} = \frac{\partial}{\partial x_i}\left(\varkappa \frac{\partial T}{\partial x_i}\right) \qquad (2.18)$$

where κ is the coefficient of heat conduction, and using (2.6) and (1.7), we obtain finally the equation for the heat inflow, applicable to the model under consideration, in the form

$$C(T; V_*)\frac{dT}{dt} = V_* \frac{\partial}{\partial x_i}\left(\varkappa \frac{\partial T}{\partial x_i}\right) - \left(p\frac{dV_1}{dV_*} + \frac{\partial \varepsilon}{\partial V_*}\right)\frac{dV_*}{dt} + \qquad (2.19)$$

$$+ V_*\left[W - \frac{1}{2G}F'(p)\frac{dp}{dt}\right]e[J_2 - F(p)]e\left[W - \frac{1}{2G}F'(p)\frac{dp}{dt}\right]$$

This equation serves for determining the temperature distribution, after solving the mechanical problem. On the right-hand side its first term determines the heat inflow associated with heat conduction. The second term determines the heat inflow associated with the dissipation of the mechanical energy, caused by irreversible volume deformation. This heat inflow equals the excess work of pressure on the plastic volume deformation over the work spent on irreversible changes in the internal energy in changing the parameter V_*. In the same way as in the earlier work [11], it can be shown that this excess must not be negative. Finally, the third term determines the heat inflow which is associated with the dissipation of the mechanical energy due to plastic shear deformation. This quantity will always be nonnegative. All this, together with the natural condition $(\partial S/\partial V_*)(dV_*/dt) \geqslant 0$, in the same way as in the earlier work [11], is in accordance with the requirements of the second law of thermodynamics that there will be no decrease in the entropy of any thermally insulated material volume of the medium.

3. Character of the dissipation in the medium and its representation by a model. The model of the medium evolved in the previous sections contains a dissipation mechanism which realizes the transformation of mechanical energy into heat. The mechanical energy losses in this model can occur, on the one hand, during plastic volume deformation, i.e. when the parameter ρ_* changes. Thereby, a part of the work performed by the compression forces to achieve volume deformation is transformed into heat (the second term on the right-hand side of Equation (2.19)). On the other hand, they will occur if there is plastic shear deformation, i.e. if $\lambda > 0$. Thereby, a part of the work of the shear stresses to achieve shear deformations will be transformed into heat (third term on the right-hand side of Equation (2.19)).

Owing to the particular structure of the basic relations of the model, which consists in their uniformity with the progress of time, the dissipation in the medium possesses properties of dry friction. Therefore, in a certain sense, the character of the motion of the medium does not depend on the velocity. To obtain a more accurate expression for what was said above, it is necessary to consider two types of motions: the first, motions in which accelerations in the equations of motion can be disregarded, and the second, motions in which the accelerations are essential (dynamic problems).

It can easily be verified that in the first case, in the absence of body forces ($F_i^e = 0$), the complete system of mechanical relations (but not the energy equation (2.19)) is invariant relative to the group of transformations

$$v_i' = kv_i / \dot{\varphi}(t), \quad p' = p, \quad \rho' = \rho, \quad \rho_*' = \rho_*, \; S_{ij}' = S_{ij}, \quad x_i' = kx_i, \quad t' = \varphi(t) \tag{3.1}$$

where $\phi(t)$ is an arbitrary monotonically increasing function and k is an arbitrary positive constant (in the presence of body forces the invariance will apply in the case $k = 1$). This means that, for the indicated class of motions of the given model, the time as such is unimportant, the character of the motion does not depend on the speed of its evolution, i.e. on the velocity of the process. In this respect the model is fully similar to the models of the theory of flow in plasticity [2-4]. It is obvious that not every model of a continuous medium in which the stresses are associated with the deformation rates possesses this property. For instance, the Newtonian model of a viscous liquid does not have this property; for the motions of a viscous medium the velocity of the process is a very considerable factor in the determination of the character of such motions.

It should be pointed out that for the case under consideration the energy equations (2.19) are, generally speaking, not invariant relative to the group (3.1). If, however, the velocities of the phenomena are such that the terms associated with the thermal conductivity of the medium in Equation (2.19) can be disregarded, this invariance will be maintained, i.e. the relation

$$T' = T \tag{3.2}$$

will supplement relation (3.1).

For a second class of motion, the complete system of mechanical relations in the absence of body forces is invariant only relative to the sub-group of similarity with respect to time and space of the group (3.1), corresponding to the particular form of the function $\phi(t) = kt$

$$v_i' = v_i, \; p' = p, \; \rho' = \rho, \; \rho_*' = \rho_*, \; S_{ij}' = S_{ij}, \; x_i' = kx_i, \; t' = kt \tag{3.3}$$

In this case too, the energy equation will not be invariant, due to the presence in it of terms which are associated with heat conduction. If these terms can be disregarded, the energy equation will also be invariant and (3.2) can supplement relations (3.3).

In the given case the invariance of the equations relative to (3.3) only allows the statement that the geometrically similar motions, i.e. motions in which the characteristics of the boundary and initial conditions are geometrically similar, will be similar also throughout the same medium on fulfilling certain simple supplementary similarity conditions.

To elucidate this a simple example is considered. Let us assume that in a space filled with the medium under consideration, which is at rest and homogeneous, there will be a detonation of an explosive charge of a spherical shape. It is required to determine the ensuing motion of the medium. In the given problem the functions sought are the radial velocity v of the particles of the medium, the two principal stresses σ_r and σ_θ and the densities ρ and ρ_*. They will depend on the radial coordinate r, the time t and the parameters: ρ_1, $\sigma_{r_1} = \sigma_{\theta_1} = -p_1$ (which are initially given), r_0 the radius of the charge, ρ_0 the density of the charge substance, q the quantity of heat released during combustion per unit mass of the charge, γ the adiabatic index of the detonation products of the charge and also a series of parameters ρ_i, K_i with dimensions of density and stress, which enter into the basic relations of the model*. It is assumed that body forces are absent. Without solving the problem, on the basis of dimensional analysis, it can be established that the dependence of the sought functions on the system of their variables and of the constant arguments can be written as

$$v = \sqrt{\frac{p_1}{\rho_1}}\, V\left(\frac{r}{r_0},\ \sqrt{\frac{p_1}{\rho_1}}\frac{t}{r_0},\ \frac{\rho_0}{\rho_1},\ \frac{q\rho_1}{p_1},\ \gamma,\ \frac{\rho_i}{\rho_1},\ \frac{K_i}{p_1}\right) \tag{3.4}$$

$$\sigma_r = p_1\Sigma_r, \qquad \sigma_\theta = p_1\Sigma_\theta, \qquad \rho = \rho_1 R, \qquad \rho_* = \rho_1 R_*$$

where V, Σ_r, Σ_θ, R, R_* are dimensionless functions of dimensionless arguments, which are written out only for V. It follows from Equations (3.4) that for the similarity of two motions occurring in one and the same medium due to two different charges, it is necessary that both charges are made of the same material (or materials with equal ρ_0, q, γ) and that the initial pressures and the density in the medium p_1, ρ_1 should be equal in both cases. In this case the functions V, Σ_r, Σ_θ, R, R_* will coincide for both motions and at geometrically similar points, i.e. at points whose coordinates are connected by the relation $r_2 = r_1 r_{20}/r_{10} \equiv \mu r_1$, the velocities, stresses and densities will be equal at instants of time which are connected by the relation: $t_2 = \mu t_1$.

The possibility of this type of similarity of motion, referred to in some cases as simple geometrical similarity, is obviously ensured by constructing relations of the model which admit the transformation (3.3) or, from another point of view, by the fact that the constant parameters in the model relations have only dimensions of density or stress.

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* The processes of heat exchange in the detonation products and between these products and the soil are disregarded.

In modeling it is necessary to bear in mind that the real motions will take place in a gravity field which, generally speaking, will influence the motion. However, in certain problems the influence of the gravity force may prove unimportant and then the usual difficulties associated with the necessity of varying the gravity acceleration forces in modeling, as is required for the analogy, can be dispensed with. As an example, the same problem relating to an explosion will be considered.

At the initial moment, due to the presence of gravity forces, initial stresses will exist in the medium, the magnitudes of which increase with depth in accordance with a linear law. After detonation of the charge, a front will propagate along the medium from its surface, which separates the area at rest from the area in which the medium is in motion. At the instant of time directly after the detonation, the area in which there is motion of the medium is relatively small and the stresses occurring therein are very considerable relative to the average initial stresses in this area as well as to their differences at the various points. Therefore, at the stage of motion under consideration, the initial stresses can be disregarded, and this means that at this initial stage the gravity forces will not have an appreciable influence. With the progress of time, the zone in which motion takes place will increase without limits and the stresses in it will drop so that at a certain stage the motion will become purely elastic. However, the equations describing the motion at this stage (the equations of the theory of elasticity) are linear and, therefore, the stress fields here will represent simply the sum of the initial field determined by the gravity forces and the field of disturbances, which is the solution of the problem whose mathematical formulation does not contain gravity forces at all. Therefore, the field of the disturbances will not depend on the gravity force. It is obvious that the velocity field in this stage will also not depend on it. Generally speaking, in the intermediate stage of motion, the gravity force will have an influence. However, it can be anticipated that this stage will not last long and the influence of the gravity force on the motion as a whole will be insignificant.

If the explosion of the charge is at a relatively small depth, the front of the disturbances will break to the surface even in the first stage of the motion, as a result of which a funnel will be formed; in this case the gravity force will have no influence on the motion of the medium or on the stress field. However, the final dimensions of the funnel can be influenced indirectly by the gravity force; for instance, in the case of considerable dimensions of the funnel a part of the material which is ejected from the funnel by the explosion will fall back into it. Experiments carried out under such conditions in various types of soft soils have shown that the final characteristics of the explosion (volume and shape of the funnel, distribution of the ground on the bank outside the funnel) and also some kinematic characteristics of the phenomenon during ejection of the soil from the funnel [14], are in agreement with the conditions of simple geometrical analogy. All these experiments show that the gravity force will influence the shape and dimensions of the funnel only in the case of very large charges, since for charges weighing up to 1000 tons this influence is small.

Further, laboratory experiments on the measurement of the kinematic characteristics of the motion of sand during explosion of small spherical charges in the absence of an open surface [15,16] also confirm the validity of a simple geometrical analogy.

Finally, our own experiments [17] on measuring directly the stress field during explosions in sandy soil, carried out under field conditions, have also shown that the conditions of simple geometrical analogy are fulfilled within the range of dimensions that have been investigated.

Thus, the existing experimental data indicate that the structure of the relations of the model intended for the description of the motion of soils should permit a simple geometrical analogy under the conditions described above, i.e. that these relations should not contain constant parameters with dimensions that cannot be expressed by the dimensions of density or stress.

The model proposed here has this property.

4. On experimental verification of the model. The experimental results mentioned in the previous section indicate that the general structure of the relations of the model considered here is appropriate. Full experimental verification of the model requires special experiments so as to establish the form of the functions – the characteristics of the medium which enter into the relations of the model. In earlier work [1], a method was proposed for experimental verification of the correctness of the plasticity condition (1.10) assumed in the model and the determination of the function $F(p)$ which is contained in this condition.

This method reduces to the following. By detonating an explosive in the ground being studied, a motion is produced which has a spherical or a cylindrical symmetry. Due to this symmetry, the orientation of the principal planes of the stress field is known, and it is possible to measure the principal stresses at various points of the moving medium by means of special stress pick-ups [strain gages]. As a result of the measurements, the complete stress tensor can be determined as a function of time and distance from the center or from the symmetry axis. Then, constructing from the measured components of the stress tensor the expressions J_2 and p in relation (1.10), which will also be functions of time and distance from the center of the charge, and excluding time as a parameter, it is possible to obtain the function F, for each of the above distances, from relation (1.10). If the function F thus constructed is equal for all distances and under various geometrical conditions, i.e. for spherical and cylindrical symmetry, this will indicate that in reality, in the case of a developed flow in the medium, the plasticity conditions of the type (1.10) are fulfilled. Simultaneously, on the basis of actual experiments, the real type of the function $F(p)$ will be determined for the soil under investigation. We refer in this case to distances that are not particularly great, at which the motion is purely elastic.

The author, jointly with V.D. Alekseenko, G.V. Rykov and A.F. Novgorodov, carried out experiments on the basis of this scheme in sandy soil during the summer of 1959, which confirmed the applicability of the condition of the type (1.10); it was established that for sandy soil under natural conditions the function $F(p)$ for the case of $p < 15$ kg/cm^2 can be written as [17]

$$F(p) = (kp + b)^2 \qquad (k,\ b = \text{const}) \tag{4.1}$$

For establishing the type of function from relations (1.2) and (1.3) (or (1.5)), the following experiment may be suggested. The specimen of soil to be investigated should be covered with a thin, easily bendable, shell which is impermeable to a liquid, and placed in a strong rigid vessel with a liquid of a known compressibility, so that the specimen is fully submerged in the liquid. By compressing the liquid in the vessel by means of a piston which is hermetic at the walls, and measuring the pressure produced in the liquid together with the displacement of the piston corresponding to this pressure, it is possible to determine from the measured results the relation between the pressure p and the density of the specimen ρ. Indeed, the piston will produce a uniform pressure in the liquid. Therefore, on the surface of the solid specimen a constant normal stress will act, which will produce in the assumed homogeneous and isotropic specimen a uniform stress field which reduces to the hydrostatic pressure p. The specimen will thus be subjected to a geometrically analogous deformation which will be homogeneous and isotropic so that, for a given pressure in the liquid, the density at all points of the specimen will be equal. Therefore, with the masses of the specimen and the liquid being fixed and the compressibility of the liquid and the deformability of the vessel under the effect of internal pressure being known, it is possible to calculate the density from the measured displacement of the piston. By making the measurements during increasing pressure, as well as during decreasing pressure, it is possible to plot the loading function (1.3) as well as the unloading function (1.2).

It should be noticed that the functions in relations (1.2) and (1.3) do not contain the characteristics of the deformation rates. Therefore, for determining these functions it is sufficient to carry out the static measurements just described. The assumption made in the model regarding the independence of the volume deformation on the deformation rate has to be additionally verified by experiments of one type or another. The experimental results considered above, which confirm fulfilment of the simple geometrical analogy, indicate that the velocity of deformation has no influence on the phenomenon as a whole and this means that, in particular, the relations describing the volume deformation will also not depend on it.

It should be pointed out, also, that the problem of the so-called "dynamic diagram" of deformation, which is sometimes discussed, has no meaning; in fact, if the diagrams of deformation under static and under dynamic conditions differ, we cannot talk about the diagram under dynamic conditions, since in this case the velocity of deformation does affect the dependence between the stresses and the deformation, i.e. under dynamic conditions we will have to deal not with a single diagram but with a family of diagrams corresponding to the various values of the invariants of the tensor of the deformation rates, which in this case will enter into relations that link the stresses with the strains.

Let us consider, also, the problem of experimental determination of the function $F(p)$ from condition (1.10) under static laboratory conditions. Firstly, the agreement of the function F thus determined with the function F determined by the dynamic method described above will be a further confirmation of the correctness of the assumption on the character of the plasticity condition. Secondly, and even more important, under static laboratory conditions the function $F(p)$ can be constructed for a wide range of values of the argument p, including very large pressures, while under the described dynamic conditions such determination is extremely difficult, since we have no means of measuring the stresses in the area which is near to the center of the explosion, where the highest stresses occur.

Under laboratory conditions, $F(p)$ can be determined from the following experiment*. Into a rigid cylinder a cylindrical specimen of the soil to

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* The experiment described here is similar to the experiment for determining the so-called compression curves in soil mechanics [8,18,19] and also experiments on the determination of the parameters of an oil-bearing stratum under elastic-plastic conditions [20]; the purpose of these experiments, however, differs.

be investigated is placed, with its diameter coinciding with the internal diameter of the cylinder and height much smaller than the diameter. By compressing the specimen, which fits tightly into the cylinder, by a piston, a uniaxial strain state will be produced in the cylindrical specimen. There will be a deviation from the uniaxial state in the neighborhood of the lateral surface of the specimen, due to the presence there of friction forces on the cylinder wall. However, since the specimen has a small diameter-to-height ratio, this deviation from the uniaxial stress state, which is localized in the small layer adjacent to the wall, can be disregarded. By measuring the displacement of the piston and the pressure under the piston (or at the bottom of the cylinder) – σ_x, we can plot the relation

$$\sigma_x = \sigma_x(\rho) \tag{4.2}$$

Using this dependence, and also relations (1.2) and (1.3), the function $F(p)$ can be plotted. If the experiment is carried out with a monotonically increasing pressure, it is sufficient to use only relation (1.3).

From (1.3) and (4.2) we can obtain J_2 as a function of ρ:

$$J_2 = \frac{3}{4}(\sigma_x + p)^2 = J_2(\rho) \tag{4.3}$$

Excluding from Equations (1.3) and (4.3) the parameter ρ, we obtain a relation between J_2 and p

$$J_2 = J_2(p) \tag{4.4}$$

At the initial stage of the experiment, when the volume and the shear deformation are both elastic, the following linear relations will apply:

$$p = A\theta, \qquad \sigma_x = B\theta, \qquad J_2 = C\theta^2 \qquad\qquad (\theta = 1 - \rho_0/\rho), \tag{4.5}$$

where ρ_0 is the initial density and A, B, C are constants which are expressed by means of the elastic constants of the medium. Therefore, at this stage relation (4.4) will coincide with the following relation derived from (4.5):

$$J_2 = \frac{C}{A^2}p^2 \tag{4.6}$$

On further increase of ρ (or θ), the limit of elasticity will be exceeded both in volume and shear deformation in either order, depending on the properties of the medium. If the expression $\theta = \theta(p)$ is substituted in the last of relations (4.5), obtained by transforming (1.3), we arrive at a relation between J_2 and p which will coincide with (4.4) when the limit of elasticity in shear is not exceeded. After exceeding this limit, these functions will differ, and from that instant onwards the function (4.4) can be considered as the desired function $F(p)$.

Similar considerations also apply to the case of the experiment with unloading from various states of compression.

Finally, another fact should be pointed out which is of paramount importance. If relation (1.3) is expressed in terms of the variables p and θ, in accordance with the model, a curve will be obtained which possesses the following properties (see Fig. 1). For small values of p, the curve will have a straight elastic section, then it will be convex upwards, will reach an inflection point and, finally, after becoming convex from below, it will rise steeply upwards. On the basis of these geometrical properties of the compression diagram of the medium, it is possible to determine the character of the changes of the profiles of the waves of stresses, velocities, etc. which occur, for instance, as a result of detonation of a spherical explosive charge in the unbounded medium.

In his paper Barenblatt [21] has studied self-similar (non-interacting) plane, one-dimensional motions of a non-linear-elastic medium, the diagram of which may possess the above-mentioned geometrical properties. It was established that if a sufficiently large constant pressure is applied to the boundary of a semi-space, a disturbance wave will propagate along the medium which is limited in the forward direction by a sharp front, i.e. a shock-wave. In the case of moderate values of this pressure, the speed of the shock-wave is relatively small so that in front of it a region of continuous motion will occur and the forward boundary of the disturbance will appear as a weak discontinuity, i.e. a characteristic. Finally, beginning from the value of the applied pressure that corresponds to the inflection point in the diagram, and even for smaller values, the disturbance will become a continuous wave, which is limited in the forward direction by the characteristic.

Considering the problem on the change of form of the disturbance caused by a concentrated detonation in the medium under consideration, it can be shown that all the qualitative types of disturbance waves described above will occur, with successive replacement of one type of wave by another. To wit, in the zone near to the charge, where the stresses are high, disturbances will propagate which are limited in the forward direction by the shock-wave. As this wave recedes from the charge, it will decay, and its velocity will decrease. At some instant its velocity will be less than the velocity of propagation of elastic disturbances which correspond to the initial section of the diagram, so that elastic motion will occur in front of the pressure-wave. Then, a time will come when the pressure-wave will vanish, the motion will become fully continuous and, finally, there will be a purely elastic wave which, with the progress of time, will recede without limits from the center of the explosion.

We have carried out special experiments for verifying the described character of the development of disturbance waves resulting from an explosion with the progress of time. The experiments fully confirmed the effects anticipated [17].

We note that in a highly idealized formulation, the problem of explosion considered here was recently studied by Zvolinskii [22]. His work also revealed all the stages of development of the disturbances described here.

Experimental confirmation of the described qualitative feature of the disturbance, which can be predicted theoretically on the basis of the geometrical properties of the diagram of volume deformation, proves that this diagram does actually possess such properties.

In conclusion, I should like to thank G.I. Barenblatt for his attention to and interest in this work.

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